Hertentamen Kansrekening – Open book

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1. Let X be a random variable taking positive values with density $f_X(x)$. Let $Y = \frac{1}{X+2}$. What is the density for Y?

   Solution: Using the transformation formula for densities, noting that the function $x \mapsto 1/(x+2)$ is bijective from the positive reals to $[0, \frac{1}{2})$ we get

   $$f_Y(y) = f_X(1/y - 2)\frac{1}{y^2}$$

2. Let X and Y be independent random variables taking values in $\{1, 2, \ldots, \}$ with

   $$P(X = i) = P(Y = i) = \frac{1}{2^i}$$

   a) Use the formula $\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}$ to check that this defines a probability weight function.

   b) Show that $P(X = Y) = \frac{1}{3}$

   c) Compute $P(\min(X, Y) \geq i)$

   Solution

   (a)

   $$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2}\frac{1}{1-\frac{1}{2}} = 1$$

   (b)

   $$\sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right)^2 = \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{3}$$

   (c) Because of independence we have the following equality

   $$P(\min(X, Y) \geq i) = P(X \geq i)P(Y \geq i)$$

   Note that

   $$P(X \geq i) = \sum_{j=i}^{\infty} \frac{1}{2^j} = \frac{1}{2^i}\sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2^i}\frac{1}{2}$$

   So the result becomes

   $$P(\min(X, Y) \geq i) = \frac{1}{4^{i-1}}$$

3. Let X, Y, Z be independent random variables, uniformly distributed on the interval $[0, 1]$.

   a) Determine the density of the random variable XY.

   b) Determine the density of the random variable XYZ.

   Solution:

   (a) For u between zero and one we have

$$P(XY \leq u) = P(X \leq u) + \int_u^1 P(Y \leq \frac{u}{x})dx$$

   Note in this decomposition that $X \leq u$ implies $XY \leq u$, no matter what value Y is taking. From this we compute

$$P(XY \leq u) = u + \int_u^1 \frac{u}{x}dx = u - u \log u$$

   From this result we get the density by differentiation

$$f_{XY}(u) = -\log u$$

   (b) For v between zero and one we have

$$P(XYZ \leq v) = P(XY \leq v) + \int_v^1 P(Z \leq \frac{v}{u})f_{XY}(u)du$$

   Next we use the result of (a) to get

$$P(XYZ \leq v) = -\log v + \int_v^1 \frac{v}{u}(-\log u)du$$

   We compute the integral with the substitution $\log u = s$ and we get

$$P(XYZ \leq v) = v - v \log v + \frac{v}{2}(\log v)^2$$

   By differentiation the density becomes from this

$$f_{XYZ}(v) = \frac{(\log v)^2}{2}$$

4. Let A_n be an increasing sequence of events. (This means $A_n \subset A_{n+1}$ for all n). Denote $A = \bigcup_{n=1}^{\infty} A_n$. Let B be an event with $P(B) > 0$.

   Show that $\lim_{n \uparrow \infty} P(A_n|B) = P(A|B)$.

Solution:

$$
\begin{aligned}
&\lim_{n\uparrow\infty} P(A_n|B)\\
&= \lim_{n\uparrow\infty} \frac{P(A_n\cap B)}{P(B)}\\
&= \frac{1}{P(B)} \lim_{n\uparrow\infty} P(A_n\cap B)\\
&= \frac{1}{P(B)} P(\cup_n(A_n\cap B))\\
&= \frac{1}{P(B)} P((\cup_n A_n)\cap B)\\
&= \frac{1}{P(B)} P(A\cap B)\\
&= P(A|B)
\end{aligned}
$$

The justification of the equality signs is as follows: 1: Definition of elementary conditional probability, 2: Compatibility of limit of a sequence with multiplication, 3: Continuity of a probability measure, applied to the probability measure P, 4: set-operation, 5: definition of event A, 5: Definition of elementary conditional probability

5. Let X and Y be independent Poisson random variables with parameters λ_X and λ_Y.

   a) What is the distribution of $X+Y$?

   b) What is the conditional distribution of X given $X+Y=k$?

   c) Can you formulate an extension of the results in a) and b) to n independent Poisson random variables?

   Solution: This exercise was already treated in the werkcolleges.

   a) Poisson with parameter $\lambda_X+\lambda_Y$

   b) Binomial with parameter $p=\frac{\lambda_X}{\lambda_X+\lambda_Y}$

   c) The sum of n independent Poisson variables with parameters λ_i is distributed according to a Poisson distribution with $\lambda=\lambda_1+\cdots+\lambda_n$.

   The conditional distribution of $(X_1,\ldots,X_n)$ given the sum $X_1+\cdots+X_n=k$ is a multinomial distribution with parameters $p_i=\frac{\lambda_i}{\lambda_1+\cdots+\lambda_n}$ and number of trials equal to k.